Symposium: Big Ideas in School Mathematics

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The "Big Ideas in School Mathematics" (BISM) is a Research Project funded by the Ministry of Education, Singapore, and administered through the Office of Educational Research, National Institute of Education, Nanyang Technological University. The project began in 2020 and its aim is to investigate various areas in relation to teaching towards mathematical Big Ideas in Singapore schools. The study has currency in so far as "Big Ideas" were introduced in the latest Syllabus Revision by the Ministry of Education. There are three sub-studies in the project: the first is on the development of instruments to measure knowledge of BISM among primary- and secondary-level students and teachers; the second is on professional development work for secondary-level teachers on BISM; the third is similar to the second but for primary-level teachers. The papers in this symposium report information and findings on all these sub-studies.

Overview of the Symposium Papers and Presenters

Presenters: Associate Professor Leong Yew Hoong (Chair), Associate Professor Toh Tin Lam (Paper 1), Mr Mohamed Jahabar Jahangeer (Paper 2), Assistant Professor Choy Ban Heng (Paper 3), Professor Berinderjeet Kaur (Paper 4)

Paper 1: Overview of the research project on Big Ideas in School Mathematics

Authors: Toh Tin Lam, Tay Eng Guan, Berinderjeet Kaur, Leong Yew Hoong, Tong Cherng Luen

This paper provides a brief overview of the entire research project and the component sub-studies.

Paper 2: Assessment of Big Ideas in School Mathematics: Exploring an Aggregated Approach

Authors: Mohamed Jahabar Jahangeer, Toh Tin Lam, Tay Eng Guan, Tong Cherng Luen

This paper reports on developments under Sub-study 1. An item from the student BISM instrument will be discussed. It argues for the use of an "aggregated approach" in considering the scores of the student responses.

Paper 3: From Inert Knowledge to Usable Knowledge: Noticing Affordances in Tasks Used for Teaching Towards Big Ideas About Proportionality

Authors: Choy Ban Heng, Yeo Boon Wooi Joseph, Leong Yew Hoong

This paper reports on developments under Sub-study 2. Part of the professional development under this project involved teachers designing their own instructional materials to foreground a targeted Big Idea. Snippets of tasks in these instructional materials will be discussed.

Paper 4: Primary School Teachers Solving Mathematical Tasks Involving the Big Idea of Equivalence

Authors: Berinderjeet Kaur, Tong Cherng Luen, Mohamed Jahabar Jahangeer

This paper reports on developments under Sub-study 3. An item from the teacher BISM instrument will be discussed. Some data on teachers' responses to the item will be shared. There are thus implications to teacher professional development on the Big Idea of Equivalence.

2023. In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives. Symposium in the proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 39–55). Newcastle: MERGA.

Overview of the Research Project on Big Ideas in School Mathematics

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Big Ideas in school mathematics can be seen as overarching concepts that occur in various mathematical topics in a syllabus. Although there has been much interest recently in the understanding of Big Ideas, there is little research done in the assessment of Big Ideas thinking. In this paper, we discuss our research on Big Ideas in School Mathematics. The study consists of three sub-studies: the first sub-study on developing an instrument to measure Big Ideas; two sub-studies on measuring students' and teachers' Big Ideas at test-points before and after a professional development on Big Ideas for primary and secondary school teachers and students.

In the recent mathematics curriculum revision conducted by the Singapore Ministry of Education (MOE), there is a new emphasis on the disciplinarity of mathematics and Big Ideas that are central to the discipline so as to bring coherence and connections between different topics. The objective of this new emphasis is to develop in students a deeper and more robust understanding of mathematics and better appreciation of mathematics (MOE, 2018; MOE, 2019). Each Big Idea connects various concepts and understanding across topics, strands and levels.

The definition of a Big Idea was proposed by Charles (2005) as "a statement of an idea that is central to the learning of mathematics, one that links numerous mathematical understandings into a coherent whole" (p.10). Prior to Charles' definition, the notion of Big Ideas in mathematics education became prominent when it was highlighted by the National Council of Teachers in Mathematics (NCTM) in 2000 that "[t]eachers need to understand the Big Ideas of mathematics and be able to represent mathematics as a coherent and connected enterprise. Their decisions and their actions in the classroom—all of which affect how well their students learn mathematics—should be based on this knowledge." (p. 17).

From our collective classroom experience, the presentation in school mathematics syllabuses as discrete strands and topics could have led teachers and students to view mathematics as a collection of topics with weak connections. Thus, Big Ideas illuminate the interconnectedness between topics across strands and this aids the robustness of understanding mathematics. The depth of understanding is dependent on the number and strength of the connections (Hiebert & Carpenter, 1992, p. 67).

Challenges in Teaching for and Measuring Big Ideas

Researchers have affirmed the existence of real challenges in the mathematics classroom for teaching Big Ideas in schools from both teachers' and students' perspectives (e.g., Hsu, Kysh, Ramage & Resek, 2007; Askew, 2013; Schoenfeld, 2019). Teachers in schools may not possess the relevant content knowledge pertaining to Big Ideas in mathematics. Lack of such knowledge is manifested in their teaching, for example, in their inability to realize that the generation of the exponent rules is traceable to the definition for positive integral exponents and that the distributive property is a Big Idea understanding for combining like terms and multiplying binomials (Hsu, Ramage & Resek, 2007).

Their deficiency of such knowledge often translates into their lack of explicit attention to Big Ideas underpinning mathematics taught in schools. Consequently, this results in students' acquisition of compartmentalized mathematical content knowledge (Askew, 2013). Lack of appropriate professional development for teachers associated with Big Ideas in mathematics, coupled with lack of time for professional development add to the challenges of teaching for Big Ideas (Hsu, Ramage & Resek, 2007; Askew, 2013).

To date, there has been little research on the assessment of Big Ideas. This could be attributed to three major reasons: firstly, researchers have different classifications of Big Ideas (e.g., Charles, 2005; Niemi et al., 2006; Singapore Ministry of Education, 2018, 2019). Secondly, the lack of clarity on the intent of the assessment. Furthermore, any additional instrument to measure Big Idea would mean an additional load to the already heavy high-stake national examinations. Thirdly, it is difficult to create items that assess thinking which link numerous mathematical understandings that cut across topics.

Conceptualization of the Research Project Big Ideas in School Mathematics

In addressing the challenges of teaching and measuring Big Ideas, a team of researchers (the authors of the papers in this symposium) conceptualized a research project Big Ideas in School Mathematics (BISM). Broadly, the aim of BISM is twofold: firstly, to develop assessment items to measure of Big Ideas in school mathematics for assessing how teachers and students connect numerous mathematical understandings into a coherent whole over multiple points of their respective developments. To date, there is a dearth of such an instrument. The second aim is to study the development of Primary and Secondary mathematics teachers' and students' knowledge of BISM across a period of time during which teachers participate in professional development about BISM. The research project consisted of three sub-studies: (1) Measures of Big Ideas in School Mathematics (BISM Measures); (2) Big Ideas in Secondary School Mathematics; and (3) Big Ideas in Primary School Mathematics.

Sub-study 1: Measures of Big Ideas in School Mathematics. This sub-study involved the development of instruments for use in sub-studies (2) and (3). The aim of this sub-study was to develop, pilot and validate instruments to measure the knowledge of Big Ideas in School Mathematics (BISM) for primary / secondary school teachers and students.

Initially, we studied the few existing instrument for the measure of Big Ideas by Niemi et al. (2006). Their items consist of three main types of tasks to measure Big Ideas in mathematics: basic computation tasks, partially-worked problems (with or without explanations), and explanation tasks. Basic computation tasks aim to assess whether students could recognize tasks representing specific Big Ideas. They could then apply the relevant Big Ideas and successfully complete the task. The designed tasks are simple and well-defined. Partially worked problems require students to fill in one to three boxes for missing numbers or symbols in the problem solution, or fill in a complete problem solving step. For an explanation task, a fully worked example is given before those partially worked examples. The selected worked example usually involves no more than 3 to 4 steps, and the fully worked examples are from similar mathematics topics but not the same topic used for assessment. The explanation tasks are based on partially worked examples with justifications. Students, in this case, need to understand the steps solved by others, and must be able to provide the principles for one of the steps. Just like the partially worked example tasks, the explanation tasks follow a fully worked example which covers a similar topic but not the same topic for real assessment.

Our approach to the assessment of Big Ideas draws on the PISA experience of assessing mathematical literacy (Stacey & Turner, 2015) in general and in Tout and Spithill's (2015) writing of items to test mathematical literacy in particular. Our overarching principle in the development and validation of items or tasks is fitness-for-purpose because because the notion of Big Ideas can be contentious at its boundaries. Also, we expected the conceptualisation of Big Ideas to be complex and cut across school mathematics content. As such, the assessment items must be accessible to students and teachers. In addition, all the assessment items are designed for computer-based testing. For details about the instrument, refer to Jahangeer et al. (2023), which occurs as a research paper in this conference proceeding.

In this study, we focused on two Big Ideas Equivalence and Proportionality. Each item, consisting of five parts, tests on only one of the two Big Ideas. Part 1 to Part 3 each consists of a selected response question focusing on the same Big Idea and are from the same topic. To facilitate thinking beyond topical content and procedural knowledge, Part 4 seeks to assist participants to look for the link connecting the three parts. Part 4 also seeks to trigger students' Big Idea concepts, if any. The participants then attempt Part 5, a question that focuses on the same Big Idea but based on a different topic. Part 5 assessed the participant's ability to transfer the knowledge of Big Idea across a different topic. We also rode on the affordance of this sub-study to address the real issue of assessment fatigue among students. This is our attempt to balance between maintaining the validity of the instrument (students must answer sufficiently many types of problems); and not over-testing the students (to avoid assessment fatigue of students, aligned to the increasing emphasis on the mental well-beings of students). This will be reported in Paper 2.

Sub-study 2: Big Ideas in Secondary School Mathematics. This sub-study aimed to study the trajectory growth in (a) secondary school teachers' knowledge of BISM in relation to their involvement in professional development related to BISM; and (b) lower secondary school students' knowledge of BISM through their two years' schooling at the lower secondary level. The findings we have obtained so far for this sub-study is presented in Paper 3.

Sub-study 3: Big Ideas in Primary School Mathematics. This sub-study is an analogue of Sub-study 2, with the focus on primary school mathematics teachers and upper primary students at Primary 5 and Primary 6. The findings we have obtained so far for this sub-study is presented in Paper 4.

The instrument developed in sub-study 1 was administered to the teacher and student participants in sub-studies 2 and 3 at various chronological points between the two years' schooling. The first test-point, administered prior to the commencement of the teachers' professional development, provided the baseline information on the state of the teachers' knowledge of BISM prior to formal participation in professional development, and the students' knowledge of BISM prior to their teachers being officially cognizant of BISM. It also guided the researchers in designing the professional development interventions for the participating teachers.

Conclusion

This study will inform how teachers understand the rationale for teaching towards Big Ideas, their belief and appreciation in the value to teach towards Big Ideas, and how these are translated into their teaching practices in their efforts to develop in students a greater awareness of the disciplinarity of mathematics, the ideas that are central to the discipline, and bring coherence and connection between different topics and across levels. In view of this, most importantly, the study will inform how students are able to better learn new mathematical knowledge with an appreciation of Mathematics as a discipline and its applications in the world.

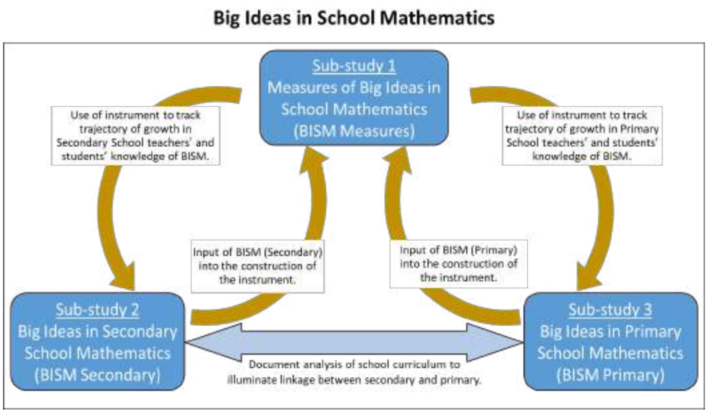

Figure 1. The relation between the three sub-studies in BISM project.

Acknowledgements

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Assessment of Big Ideas in School Mathematics: Exploring an Aggregated Approach

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In this paper we report our development of instruments to measure Big Ideas in school mathematics. In tackling the issue of assessment fatigue among students, we present an aggregated approach to measure students' knowledge of Big Ideas.

There has been little research done on how the knowledge and understanding of Big Ideas can be assessed. In one of the rare examples we could find, Niemi et al. (2006) suggested three main types of assessments to measure Big Ideas in mathematics: basic computation tasks, partially-worked problems (with or without explanations), and explanation tasks. Charles' (2005) definition of a Big Idea in Mathematics as "a statement of an idea that is central to the learning of mathematics, one that links numerous mathematical understandings into a coherent whole" (p.10) implies the need to contrast a task across more than one topic to be able to tease out the use of a Big Idea in the task. We followed this basic principle in constructing an instrument to assess Big Idea 'ability'. An example of an item, consisting of five parts, on Equivalence is shown in Figure 1. We have piloted some of the items which we have constructed. The dimensionality of these items are reported in Jahangeer et al. (2023), a separate paper in this conference. An important consequence from a Rasch analysis was that we could only use Part 5 as a reliable measure of Big Idea 'ability' since within an item, Parts 1 to 3 violate the item independence requirement of a Rasch scale.

Assessment Fatigue

Assessment has always been an integral part of teaching and learning. Analysis of assessment performance is used for a variety of purposes including placement, selection and certification. In many countries, standardised and high stakes assessments are put in place at milestone grades to determine placement and selection of students to the next course of their education. Well-designed assessment tools and analysis can provide accurate information regarding student learning.

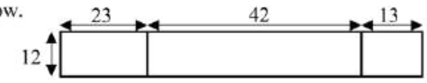
Inaccuracies or deviations from what students have mastered could have been contributed by the students themselves. In particular, the cognitive demand required on students may contribute to them experiencing cognitive fatigue, which naturally affects their overall performance. According to Ackerman and Kanfer (2009), "[a]nticipations of subjective fatigue may lead some individuals to avoid tasks altogether" (p. 176). The duration of an assessment may result in unwilling students not committed to performing to the best of their abilities, affecting the validity of the responses. Thus, a balance between the reliability and validity of the assessment and the duration of assessment without causing a negative anticipation of cognitive fatigue, is an area of worthwhile concern for educators and researchers.

Returning to our attempt to assess Big Idea 'ability', the same consideration of duration of assessment in relation to test validity and reliability arises. Each item of ours necessarily consists of parts to enable a Big Idea to surface across different topics. However, just two items would require at least 30 minutes. A valid Rasch scale would require at least six items to cover a significant range of ability. We derived this based on Andrich's work which, when describing the invariance of appropriate comparison on measures using the Rasch model, used a six-item questionnaire for an example (Andrich, 1988, p. 22).

Part 1: The diagram below shows three rectangles joined together to form a bigger rectangle.

You may use the diagram to fill in the blanks below.

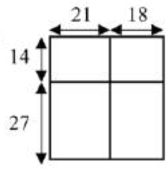
$23 \times 12 + 42 \times 12 + 13 \times 12 = ? \times 12$



Part 2: The diagram below shows four rectangles joined together to form a bigger rectangle.

Fill in the missing numbers below.

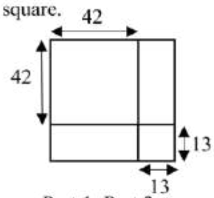
$14 \times 21 + \Delta \times 18 + 27 \times 21 + 27 \times 18 = \heartsuit \times 39$



Part 3: The diagram below shows 2 squares and 2 rectangles joined together to form a bigger square.

Fill in the missing numbers below.

$55 \times 55 = 42 \times 42 + 13 \times 13 + 13 \times \Delta \times \heartsuit$



Part 4: Which of these following statements best describes the common mathematical idea across Part 1, Part 2 and Part 3?

a) I used diagrams for the parts

b) I used Equivalence for the parts

c) I used Guess and Check for the parts

d) I used Proportionality for the parts

e) Others (Please elaborate)

Part 5: The shaded area of the figure below can be used to show a mathematical statement. Which of the following statements matches the shaded part of the figure?

a) $(1 + 6) + (2 + 5) + (3 + 4) \ldots + (6 + 1) = 6 \times 7$

b) $1 + 2 + 3 + \ldots + 6 = (6 \times 7) \div 2$

c) $1 + 2 + 3 + \ldots + 7 = (7 \times 8) \div 2$

d) $1 + 2 + 3 + \ldots + 8 = (8 \times 9) \div 2$

e) $1 + 2 + 3 + \ldots + 7 = (7 \times 8)$

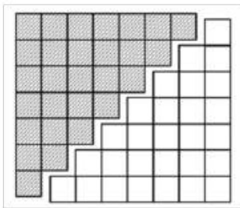


Figure 1. An item comprising 5 parts.

An Aggregated Solution

We base our solution to the conundrum on the methodology and raison d'etre of sampling, i.e., to understand a population, there is no need for every student to complete the entire instrument. For example, the Programme for International Student Assessment (PISA) carried out international standardized testing every three years across various domains. Each domain consists of items which are subdivided into smaller blocks and each student involved in the assessment will be given a booklet made up of a few blocks. PISA made use of 'plausible values' to determine a student's performance and to give a population score instead of an individual score. The successful computation of plausible values, however, requires a deeper knowledge of mathematics which is not accessible to educators, generally. We are proposing a simpler structure that is mathematically easier and can be implemented by educators in schools.

We propose an aggregated structure which involves the creation of a 'Super-Student' (SS). Each SS is made up of four students of similar ability in Mathematics. A random grouping of students to form a SS will likely confound the results—a strong student in the grouping could have solved a difficult item and a weaker student in the grouping could not solve an easy item assigned. Thus, the SS would be invalid due to the misfit in responses. Although no study has been done to assess the correlation between mathematics ability and big idea thinking, Schoenfeld (2019) mentioned in his study that high performing individuals are able to see and use Big Ideas in problem solving. We thus make a reasonable assumption that students of similar ability may have the same level of Big Idea thinking. In this light, we propose to constitute an SS, with all four students having identical ability (ideally but impossible in practice), by rank ordering students based on their past semestral assessment marks as a proxy of their mathematical ability. Going down the list, every four students are grouped into an SS and given a new SS ID. For example, in a school of 320 students, the top four students will constitute SS01, the next four SS02 and the last four students in the ordered list will be SS80. Triangulation can be carried out with teachers to validate that the students grouped together are indeed of similar ability. To differentiate the students within each SS, a suffix is added, e.g., SS01a, SS01b, SS01c and SS01d for the four students that constitute SS01. This is done to facilitate the correct distribution of the items.

We envisage a final instrument for a Big Idea consisting of eight items (each with five parts). The eight items are split into eight testlets, T1 to T8 as shown in Table 1. Each testlet is only made up of two items and each student attempts only one of the testlets. Table 1 shows how the testlets are distributed to the students as well as to each SS. Since each testlet has only two complete items, it can be administered easily within a much shorter duration and will reduce cognitive fatigue.

Table 1

Matrix Distribution of Items to Two SS Comprising a Total of Eight Students

	T1	T2	T3	T4	T5	T6	T7	T8
I1	SS01a							SS02d
I2	SS01a	SS02a						
I3		SS02a	SS01b					
I4			SS01b	SS02b				
I5				SS02b	SS01c			
I6					SS01c	SS02c		
I7						SS02c	SS01d	
I8							SS01d	SS02d

As a result of the SS structure and distribution of testlets, each SS will have taken the entire set of items while each student only attempts two items. Thus, the duration required to complete the test is only 25% of the time required to complete all the eight items. The score collated will be for each SS instead of for every student in the school. This SS structure can be used not only in obtaining an aggregated score for assessing group performance on an instrument, but it can also be used for validating an instrument during its initial item creation stage.

Conclusion

While assessments are important to monitor learning, too many high stakes assessments will reduce available time for teaching, erode the joy of learning and cause a high level of worry and stress about exams and results. However, assessment remains crucial to monitor if learning has taken place and is an important feedback mechanism to improve teaching as well as learning. In place of high stakes assessments, an aggregated structure as proposed may gather sufficient information regarding learning without increasing student cognitive load nor take up too much precious curriculum time. This may be a worthwhile contribution towards the joy of learning.

One of the main issues that arise from the SS structure is the validity of the SS itself. How similar are the four students within each SS? With no prior research done on the relationship between math ability and Big Idea thinking, it is difficult to validate the structure we have proposed. At this juncture, we have piloted the items and the SS structure is due to be tested and analysed later. We intend to explore and analyse the performance of the SS using two different approaches.

The first approach is to study the misfit of SS scores using Rasch analysis. In the development of the instrument, the items would be calibrated and validated using Rasch model. Using the same Rasch model analysis, we will be able to do a fit analysis by looking at person (SS) misfit information, if any. In the event of any person misfit cases, we hope that the misfit is due to the individual students doing the two items erratically, and not caused by the different students within the SS, e.g., the misfit is due to SS01a getting items with higher difficulty correctly while SS01c answering items with lower difficulty incorrectly. The second approach is by comparing a super-student score against the scores of each of the four students forming the super-student structure based on plausible values created for each student. The technique to calculate the plausible values can be found in Von Davier et al. (2009). We will collect our data from July 2023 and report the results thereafter.

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From Inert Knowledge to Usable Knowledge: Noticing Affordances in Tasks Used for Teaching Towards Big Ideas About Proportionality

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Teaching towards big ideas provide opportunities for teachers to think deeply about content and pedagogy in order to support their students to see connections in mathematics. However, teachers may not always activate or mobilise their knowledge in classroom situations. This paper looks into how a teacher, Peter, think about the tasks in his instructional materials he crafted to uncover what he may notice about the affordances of the tasks for teaching proportionality.

Teaching towards big ideas, a recent initiative included in the 2020 Singapore Mathematics Syllabus (Ministry of Education-Singapore, 2019), provides opportunities for teachers to think more deeply about what and how they teach in order to support their students to see connections in mathematics (Choy, 2019). Doing this requires teachers to pay attention to the mathematics embedded in the curriculum, discern the details of the big ideas, and perceive the affordances in tasks for bringing out these ideas (Choy, 2019). A key enabler is the mathematical knowledge for teaching (Ball et al., 2008) that teachers can activate during classroom instruction. This suggests a key distinction between inert knowledge (Renkl et al., 2010) and usable knowledge, or what they mobilise during teaching. Kersting et al. (2012) hypothesized that "teachers with more usable knowledge are able to apply that knowledge to the design and improvement of instruction in their classrooms" (p. 573). Furthermore, as Choy and Dindyal (2021) had pointed out, it is not trivial for teachers to notice the affordances of tasks and harness them to improve instruction. Here, we explore how teachers can be supported, through professional learning (PL) sessions, to transform their inert knowledge into usable knowledge through the discussion and design of instructional materials. This paper is guided by the following research question:

- How does a PL session that focuses on the design of instructional materials activate his inert knowledge of a big idea in mathematics?

Contexts and Methods

The six teacher participants in the study reported here is part of a larger project on "Big Ideas in School Mathematics", which focused on the notion of teaching towards big ideas in Singapore. These six teachers participated in a series of professional learning (PL) sessions to unpack big ideas about proportionality (Yeo, 2019) so that they can design instructional materials and lessons for teaching the topic of ratio and rates in Secondary One. In the first session, the second author discussed the idea of proportionality from a few perspectives: when one quantity is multiplied by n, the other quantity is also multiplied by n (which we will call proportional reasoning), the equality of two ratios (e.g. $\frac{y_2}{y_1} = \frac{x_2}{x_1}$ for direct proportion), the rate $\frac{y}{x}$ is constant for direct proportion, and the product xy is constant for inverse proportion. Two main approaches to solving problems involving proportionality were shared: proportional reasoning via the unitary method and using the constant rate $\frac{y}{x}$ directly. In the next two sessions, the second and third authors facilitated discussions on the use of these two approaches, as well as others (Weinberg, 2002), to solve problems involving constant rates and supported the teachers in thinking about the design of instructional materials to incorporate proportionality in questions involving ratio, percentage, currency exchange and speed. Of interest in this paper is the instructional material shared by Peter (pseudonym), one of the teachers, during the fourth PL session, which was facilitated by the first author. Data collected include video and voice recordings of the PL session, and the draft instructional material designed by Peter. For this paper, the findings were generated from Peter's sharing on his thinking behind the design of the instructional material used for teaching rate, as well as the interactions between him and the other teachers in the PL session. Analyses were guided by the following questions:

- What knowledge on proportionality did Peter utilise in his design?
- What inert knowledge on proportionality did Peter activate during the PL session?

Three Short Snippets of Peter's Thinking

In this section, we begin by describing three short snippets of Peter's thinking, juxtaposed with what the other teachers said in response to the questions or prompts by the first author (BH). We then unpack Peter's thinking behind his design or choice of tasks put into instructional material before we characterise his understanding of proportionality in terms of what he knew inertly (Renkl et al., 2010) and what he was able to access and use—usable knowledge (Kersting et al., 2012)—through his interactions during the PL session.

Snippet 1: Shampoo Investigation Task

Peter began by describing the investigation task he placed at the beginning of the instructional material (see Figure 1). He had wanted the students to rely on their intuition and explain how they solve the problem *before* teaching them about the concept of rate.

Investigation!

- Which bottle of shampoo is more **value for money** (cheaper for the same volume of shampoo)?
- How do you determine that?

Bottle A	Bottle B
	
400 mL: $5.00	500 mL: $6.00

Figure 1. Shampoo problem.

When asked about how students might respond to the task, Peter responded that "some of them might choose to ignore the idea of same volume and just superficially choose the cheapest" [38:26]. Teacher M then shared that "they would use the unitary method" to obtain the cost of shampoo for 100 mL and subsequently 1 mL [38:52]. With more prompting, Peter highlighted that students could "change the volume to 2 litres" [39: 47] and compare. Teacher N also offered a similar size-change strategy (Weinberg, 2002) by changing the price to $30. Building on this discussion, the first author highlighted that these different methods (without using rate explicitly) were all based on the overarching idea of proportionality.

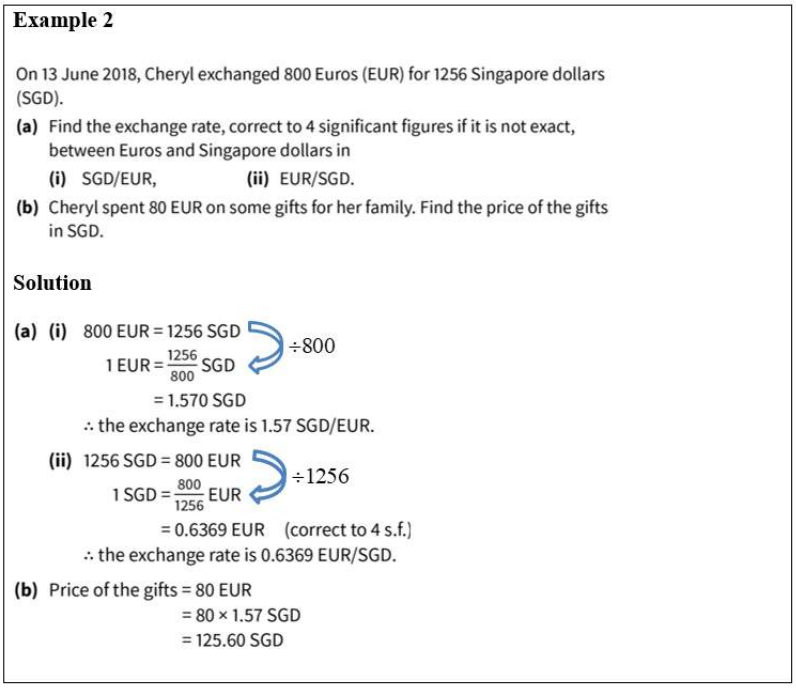
Snippet 2: Fastest Typist Problem

After the investigative task, Peter defined rate as "a quantity per (one) unit of another quantity" and selected a series of tasks, meant for students to compute rates in his instructional material. One such task is given as follows: Jayden can type 720 words in 6 minutes, Ithiel can type 828 words in 18 minutes and Zhi Rui can type 798 words in 19 minutes. Who is the fastest typist?

Peter had intended the task to be used merely for computation. At this juncture, the first author highlighted the possibility of "looking more closely at the numbers used" and modify the numbers to bring out the idea of proportionality more explicitly. The first author suggested Peter to consider how the numbers can be changed to provide opportunities for students to exercise their proportional reasoning. In addition, he highlighted to Peter that the current set of numbers did not require students to do deliberate calculation using "proportional reasoning"; instead, students would just need to mentally estimate that Jayden has to be the fastest typist because he could type around 700 words within 6 minutes, as compared to what the other two could type in a much longer time (18 or 19 minutes). Of course, students could have multiplied 720 by 3 (proportional reasoning) to compare Jayden's typing speed against the other two. Through the discussion, Peter became aware of how the item could be used to emphasise different aspects of proportionality.

Snippet 3: Exchange Rate Problem

The rest of the instructional material focused on providing opportunities for students to calculate per (one) unit rates instead of looking out for opportunities to highlight the "power of proportionality" to make sense of comparisons between two quantities. For instance, Peter went through Example 2 (See Figure 2) as merely computational without noticing the alternative solution to part (b) of the question. When the first author prompted the teachers to look more closely at the answer to part (b), Peter realised that students could simply divide 1256 SGD (given in the stem of the question) by 10 using the idea of proportionality.



Example 2

On 13 June 2018, Cheryl exchanged 800 Euros (EUR) for 1256 Singapore dollars (SGD).

(a) Find the exchange rate, correct to 4 significant figures if it is not exact, between Euros and Singapore dollars in

(i) SGD/EUR, **(ii)** EUR/SGD.

(b) Cheryl spent 80 EUR on some gifts for her family. Find the price of the gifts in SGD.

Solution

(a) (i) 800 EUR = 1256 SGD

1 EUR = $\frac{1256}{800}$ SGD ÷800

= 1.570 SGD

∴ the exchange rate is 1.57 SGD/EUR.

(ii) 1256 SGD = 800 EUR

1 SGD = $\frac{800}{1256}$ EUR ÷1256

= 0.6369 EUR (correct to 4 s.f.)

∴ the exchange rate is 0.6369 EUR/SGD.

(b) Price of the gifts = 80 EUR

= 80 × 1.57 SGD

= 125.60 SGD

Figure 2. Exchange rate problem.

Discussion

Taken together, the three snippets detailed in this short paper suggest that while Peter and the other teachers were aware of the ideas of proportionality (as seen in Snippet 1), he might not always be able to *notice* these ideas and *harness the affordances* of the tasks embedded in the instructional materials he had designed (Choy & Dindyal, 2021). As seen from the three snippets, he was able to articulate the knowledge about teaching proportionality, especially the idea of providing opportunities for students to reason proportionally using different solution strategies (Weinberg, 2002). Yet, he did not always notice affordances of these tasks to bring out the idea of proportionality and instead focused on emphasising a fixed way of finding rate and solving missing value questions. In other words, it is not trivial for teachers to activate their inert knowledge about teaching proportionality to generate usable knowledge that can potentially enhance students' understanding of this big idea when designing instruction materials. What matters is not simply what the teachers know, but how they can learn to mobilise their knowledge in actual classroom situations (Ball et al., 2008; Kersting et al., 2012).

These snippets not only highlight the complex and perennial issue of knowledge activation in the act of teaching but also provide insights into how professional learning activities can be structured to support teachers to bridge the gap between their knowledge and classroom practice. First, such professional learning can be structured around discussion of lessons and more specifically, the design of instructional materials. Designing lesson materials provide an avenue for teachers to transform their knowledge into something usable, and hence enhance the possibility of them mobilising their inert knowledge. Second, we see the need for teachers to learn to notice affordances for using tasks and other instructional materials because doing this provides opportunities for teachers to generate new possibilities that can potentially change practices. Lastly, the role of a knowledge facilitator to support teachers to notice new possibilities in their design of instructional materials, in the context of professional learning sessions, should not be underestimated. How such sessions could be facilitated remains under-studied and could be a fruitful area for future research.

Acknowledgements

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Primary School Teachers Solving Mathematical Tasks Involving the Big Idea of Equivalence

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The primary school mathematics syllabuses in Singapore as of the year 2020 reinforces that Big ideas are central to the learning of mathematics. In support of the push to teach for big ideas, a research study is presently underway. A part of it is on the professional development (PD) of primary school mathematics teachers. As part of the PD teachers attempted a mathematical task as measure of the big idea, Equivalence, in an online environment at the start of their PD. Data from the task show that teachers, were generally not cognisant of the big idea of equivalence when solving the task. They were also unable to distinguish between a heuristic (diagrams) and a mathematical idea about relationships, specifically equivalence as in the mathematical task.

The revised school mathematics curriculum, in Singapore, as of 2021 has placed emphasis on learning mathematics as a body of connected knowledge (Ministry of Education, 2019). Four themes, namely properties and relationships, representations and communications, operations and algorithms, and abstractions and applications together with six big ideas have been emphasised for the teaching of mathematics in primary schools. A "big idea is a statement of an idea that is central to the learning of mathematics, one that links numerous mathematical understandings into a coherent whole" (Charles, 2005, p. 10). The six big ideas are diagrams, equivalence, invariance, measures, notations, and proportionality. A research study, Big Ideas in School Mathematics (BISM) is presently underway in Singapore and a part of it is on professional development (PD) of primary school mathematics teachers related to the enactment of Big Ideas in their mathematics instruction. Research has documented that teachers' lack of relevant content knowledge of Big Ideas in mathematics translates into their lack of explicit attention to Big Ideas underpinning mathematics taught in schools and results in developing isolated compartments of mathematical knowledge in their students (Askew, 2013). The study reported in this paper draws on part of the data from the BISM project. It attempts to uncover if teachers drew on the big idea of equivalence when solving mathematical tasks that encompass equivalent relationships at the beginning of their PD.

The Study

Participants and Instrument

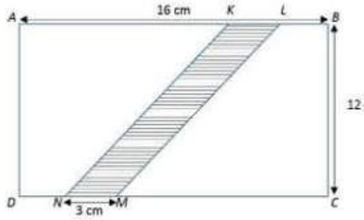
All the mathematics teachers in two primary schools, P1 and P2, participated in the PD (see Kaur et al. 2021; 2022). The PD was spread over two years. In the first year 24 teachers from school P1 and 32 teachers from school P2 and in the second year 23 teachers from school P1 and 33 teachers from school P2 participated in the PD. Due to teacher movement in and out of schools, in the second year there was one less teacher in school P1 and one more teacher in school P2.

Each year during the first session of the PD teachers attempted a set of three mathematical tasks in an online computer environment. These tasks were part of a collection of tasks that were being put together as measures of two big ideas, namely equivalence and proportionality. In the first-year teachers attempted 2 tasks on proportionality and 1 on equivalence, and in the second year they attempted 1 task on proportionality and 2 tasks on equivalence. We limit the data in this paper to the item on equivalence that teachers in School P1 attempted during the first session of their PD in the first year.

Figure 1 shows the equivalence task the teachers attempted in the first session of their first year. The task had 5 parts. Parts 1, 2 and 3 were tasks independent of each other that involved geometrical shapes and measurement. Similar tasks are found in end of school examinations for primary 6 in Singapore schools. Part 4-1 prompted the teachers to review their solutions to Parts 1, 2 and 3 and reflect on any common idea they may have drawn on whilst working on their solutions. Part 4-2 offered some options for teachers to consider about what may have guided their solutions in Parts 1, 2 and 3. Part 5-1 was yet another task on geometry and measurement that teachers had to attempt. Following Part 5-1 was Part 5-2, where teachers were again asked to review their solutions for Parts 1, 2, 3 and 5-1 and consider what may have guided their solution process.

Part 1

The figure below shows a rectangle $ABCD$.
The lines KN and LM are parallel.

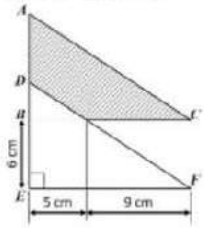


Find the shaded area.

• 18 cm² • 36 cm² • 156 cm² • 192 cm² • Others

Part 2

In the diagram below, ABC and DEF are two identical right-angled triangles. Triangle DEF is partly placed on top of triangle ABC.

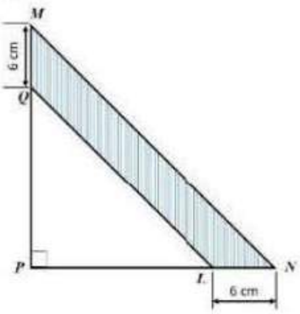


Find the shaded area.

• 42 cm² • 57 cm² • 75 cm² • 84 cm² • Others

Part 3

In the diagram QPL and MPN are two right-angled isosceles triangles. QPL is placed on top of MPN such that $QM = LN = 6$ cm. Area of the shaded part is 144 cm².



Find the length of PL.

• 21 cm • 24 cm • 27 cm • 30 cm • Others

Part 4-1

What is a common mathematical idea in Part 1, Part 2, and Part 3?

To help you recall click on the links below to see the parts		
Part 1	Part 2	Part 3

Enter your response in the space below.

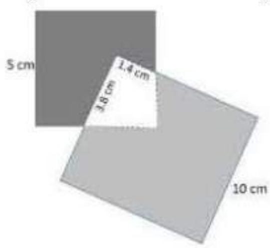
Part 4-2

Which of the following statements best describes a common mathematical idea across Part 1, Part 2 and Part 3?

- I used diagrams for the parts.
- I used equivalence for the parts.
- I used guess and check for the parts.
- I used proportionality for the parts.
- Others (Please elaborate)

Part 5-1

The diagram shows a lighter square with side 10 cm placed on top of part of a darker square with side of length 5cm. Their common region is unshaded.



Find the difference between the area of the lighter region and the area of the darker region? (You may want to consider drawing a model)

• 5 cm² • 5.2 cm² • 5.32 cm² • 70 cm² • Others

Part 5-2

Choose one of the statements that best describes what is common about Part 1, Part 2, Part 3 and Part 5?

To help you recall click on the links below to see the parts			
Part 1	Part 2	Part 3	Part 5-1

- In all these parts I used diagrams.
- In all these parts I used equivalence.
- In all these parts I used guess and check.
- In all these parts I used proportionality.
- Others (Please elaborate)

Figure 1. Example of mathematical task illuminating equivalence as a big idea.

Data and Discussion

Table 1 shows the performance of 24 teachers from School 1 on the mathematical item shown in Figure 1.

Table 1

Performance of Teachers on Mathematical Task Shown in Figure 1

Task	Response	n (%)
Part 1	36 cm^2 (correct answer)	21 (87.5)
Part 2	57 cm^2 (correct answer)	18 (75.0)
Part 3	21 cm (correct answer)	18 (75.0)
Part 4-1	Others*	24 (100)
Part 4-2	I used diagrams for the parts.	7 (29.2)
	I used equivalence for the parts.	3 (12.5)
	I used guess and check for the parts.	2 (8.3)
	I used proportionality for the parts.	9 (37.5)
	Others (Please elaborate)	3 (12.5)
	Use algebra / Cut-outs and diagrams / Use algebra and part-whole relations	
Part 5-1	Others (75 cm^2–correct answer)	7 (29.2)
Part 5-2	In all these parts I used diagrams.	7 (29.2)
	In all these parts I used equivalence.	3 (12.5)
	In all these parts I used guess and check.	2 (8.3)
	In all these parts I used proportionality.	12 (50.0)
	Others (Please elaborate)	0 (0.0)

*Responses of the teachers were coherent with Part 4-2.

It is apparent from the data in Table 1 that at least 18 (75%) of the teachers managed to work through Parts 1, 2 and 3 of the task and arrive at the correct answer. 12 of them mentioned using diagrams, equivalence and part-whole relations as mathematical ideas in their solutions. To resolve Part 1, as shown in Figure 2, one may find the area of the shaded portion by finding the difference between the areas of rectangles with sides 16 cm by 12 cm and 13 cm by 12 cm. Similarly for Parts 2 and 3, teachers may have 'used diagrams' to illuminate relationships. It appears that some teachers were using diagrams as a heuristic to illuminate a mathematical idea which many failed to name as equivalence. This may have been due to a lack of 'vocabulary' in their mathematics discourse.

However, for Part 5-1 it appears that teachers were challenged when trying to construct a relationship using diagrams. The hint provided could have led them to make equations such as:

- area of lighter region + area of overlap = 100 cm^2
- area of darker region + area of overlap = 25 cm^2

and observe a relationship, but many appear to have failed at it. It is not clear what teachers meant by 'used proportionality' in their responses to Parts 4-1, 4-2 and 5-2. As teachers were not interviewed about their responses to the parts of the task, we are unable to decipher what they meant by this.

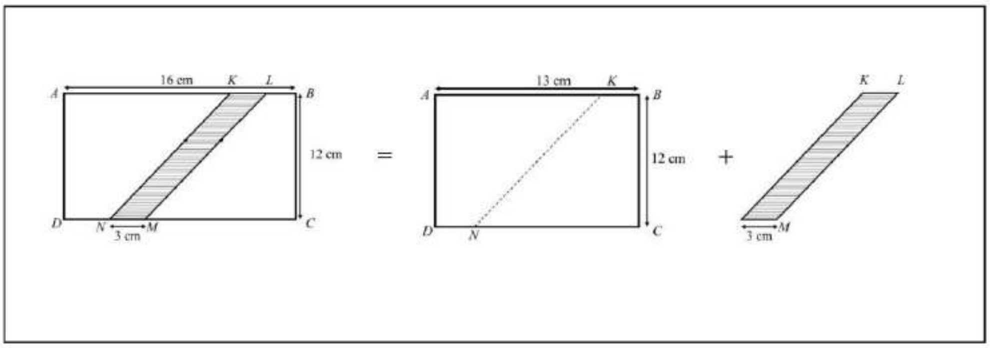


Figure 2. Equivalent relationship of parts in Part 1 of task.

Conclusion

It is apparent from the teachers' responses to the parts in Figure 1 that generally they were not cognisant of the big idea of equivalence which is stated as follows in the mathematics syllabus for primary schools (Ministry of Education, 2019, p. 15):

> Equivalence is a relationship that expresses the 'equality' of two mathematical objects that may be represented in two different forms. The conversion from one form to anther equivalent form is the basis of many manipulations for analysing, comparing, and finding solutions. In every statement about equivalence, there is a mathematical object (e.g. a number, an expression or an equation) and an equivalence criterion (e.g. value(s) or part-whole relationships).

The findings of the study reported here were critical in shaping the following PD sessions as teachers' lack of relevant knowledge of Big Ideas translates into their lack of explicit attention to them in their instruction (Askew, 2013). During the second session of the PD, teachers shared how they had attempted to resolve Parts 1, 2 3, and 5-1. The whole group discourse together with inputs from the experts (University professors) created a shared vocabulary for Big Ideas and specifically—equivalence and how such an idea facilitated solutions of mathematical tasks similar to the ones in Figure 1 and others in the school mathematics curriculum.

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